

Exponents for the number of pairs of nearly favorite points of simple random walk in \mathbb{Z}^2

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Abstract

We consider the problem on nearly favorite points of simple random walk in \mathbb{Z}^2 which Dembo et al. suggested. We determine the power exponents for the numbers of pairs of α -favorite points. These are conjectured to coincide with the exponents of the corresponding quantity for late points and the one for high points of the Gaussian free field for which their exact values are known. Our result verifies in almost sure sense. We also estimate this value in average and obtain the coincidence of it with the corresponding ones for the Gaussian free field.

1 Introduction, known results and unsolved problems

In this paper, we study a property of the local time of a random walk and special sites among the random walk range; especially we solve the open problem posed in [10]. Although the topic is classical, many questions related to it are recently propounded and investigated, many of them remaining to be cleared up. About fifty years ago, Erdős and Taylor [16] proposed a problem concerning simple random walk in \mathbb{Z}^d : how many times does the random walk revisit the most frequently visited site (up to a specific time)? They computed certain upper and lower bounds for this number and gave a conjecture on the exact asymptotic form of it. Forty years later, Dembo et al. [8] positively solved their conjecture and raised additional open problems concerning this point. We call the most frequently visited site among all the points of the range (of the walk of finite length) a favorite point. There are a few known results concerning geometric structure of the favorite points. For example, in [19] we showed that the favorite point of the simple random in \mathbb{Z}^d with $d \geq 2$ does not appear in the inner boundary from some time on a.s. Lifshits and Shi [18] verified that the favorite point of 1-dimensional random walk tends to go far from the origin. Additional open problems concerning geometric structure of the favorite points are raised by Erdős and Révész [14, 15] and Shi and Tóth [21] but almost no definite solution to them is known for multi-dimensional walks.

In this paper, we deal with nearly favorite point, whose local time is close to that of favorite point. We define the set of α -favorite points $\Psi_n(\alpha)$ for $0 < \alpha < 1$ as in [10] (see (3) in the next section) and obtain that for any $0 < \alpha, \beta < 1$ with probability one

$$\lim_{n \rightarrow \infty} \frac{\log |\{(x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^\beta\}|}{\log n} = \rho_2(\alpha, \beta) \quad (1)$$

and

$$\lim_{n \rightarrow \infty} \frac{\log E[|\{(x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^\beta\}|]}{\log n} = \hat{\rho}_2(\alpha, \beta), \quad (2)$$

for some $\rho_2(\alpha, \beta)$, $\hat{\rho}_2(\alpha, \beta)$ which will be introduced later. In [10], determining the value of $\rho_2(\alpha, \beta)$ is proposed as an open problem. The present study of the α -favorite points is motivated by the desire to understand the structure of the random field of a local time as a configuration of nearly favorite points. In addition, we want to know the relation between the local time of a random walk and the Gaussian free field (GFF). In fact, there are several related known results. For $0 < \alpha < 1$, the known results study the set of α -high points of the GFF in $\mathbb{Z}_n^2 (= \mathbb{Z}^2/n\mathbb{Z}^2)$ (sites where GFF takes high values) and

α -late points in \mathbb{Z}_n^2 (sites where the hitting time of the random walk is large). Daviaud [4] estimated the number of α -high points of the GFF in the same forms as (1) and (2). For α -late points in \mathbb{Z}_n^2 , [10] obtained the result corresponding to (1) and Brummelhuis and Hilhorst [3] that corresponding to (2). In each case, the exponent coincides with that of (1) or (2). [13] find out some relations between the local time and the GFF, and according to them together with the results mentioned right above we have expected our results to hold true.

We explain the outline of the proof of main results of this paper. The proof of the lower bound in (1) relies on the moment method which is used by [7, 9, 10]. Now, we explain one of the reason why the lower bound in Theorem 1.1 in [8] has been unsolved for a long time and the method of the proof in [8, 20], which is essentially the same as that of [7, 8, 9, 10]. If we use the Paley-Zygmund inequality for $|\Psi_n(\alpha)|$, we need to estimate the second moment of it. However, it is much larger than the square of the first moment and, hence, it is impossible to use directly the moment method for $|\Psi_n(\alpha)|$. Then, to estimate the lower bound, they consider the set of points $\Psi'_n(\alpha)$ called succesful points which is a slightly smaller set than $\Psi_n(\alpha)$. (For example, see (3.7) in [20].) In fact, the second moment of $|\Psi'_n(\alpha)|$ is of almost same order as the square of the first moment of $|\Psi'_n(\alpha)|$ and the first moment of $|\Psi'_n(\alpha)|$ is same order as that of $|\Psi_n(\alpha)|$. Then, they use the moment method for $|\Psi'_n(\alpha)|$, and, hence, the definition of $\Psi'_n(\alpha)$ naturally yields the lower bound of $|\Psi_n(\alpha)|$. In this paper, to use this method, we choose a proper set corresponding to $|\Psi'_n(\alpha)|$.

The proof of the upper bound in (1) is based on the method developed in [7, 10]. In [7], they choose many small circles and consider the probability that points in the circle is covered by a random walk under the condition on a crossing number around the circle. In [10], this method is used to study late points. Unlike in [7, 10], we consider the probability that the point in the circle is the favorite point conditioned crossing number around the circle in Lemma 5.1. After estimating this probability, we follow the argument in [10]. The result of Lemma 5.1 corresponds to Lemma 6.1 in [10] and after having Lemma 5.1, the proof is almost the same as the proof of the upper bound in [10]. On the other hand, the method of the proof of Lemma 5.1 is different from that of Lemma 6.1 in [10]. We estimate the upper bound of the probability of the intersection of the events that two points are the favorite points and the crossing number around them is conditioned.

In the proof of (2), we estimate the probability that two points whose distance is specialized are favorite points. In particular, to show the lower bound in (2), we use a certain general Markov chain rule studied in subsection 3.2.

2 Framework and main results

To state main results, we prepare notations. Let d be the Euclidean distance. Let $D(x, r) := \{y \in \mathbb{Z}^2 : d(x, y) \leq r\}$ and for any $G \subset \mathbb{Z}^2$, $\partial G := \{y \in G^c : d(x, y) = 1 \text{ for some } x \in G\}$. For $x \in \mathbb{Z}^2$, we sometimes omit $\{\}$ in writing one point set $\{x\}$. Let $\{S_k\}_{k=1}^\infty$ be a simple random walk on the 2-dimensional square lattice. Let P^x denote the probability for a simple random walk starting at x . We simply write P for P^0 . Let $K(n, x)$ be the number of times the simple random walk visits x up to time n , that is, $K(n, x) = \sum_{i=0}^n 1_{\{S_i=x\}}$. For any $D \subset \mathbb{Z}^2$, let $T_D := \inf\{m \geq 1 : S_m \in D\}$. For simplicity, we write T_{x_1, \dots, x_j} for $T_{\{x_1, \dots, x_j\}}$. Let $\tau_n := \inf\{m \geq 0 : S_m \in \partial D(0, n)\}$. In addition, $[a]$ denotes the smallest integer n with $n \geq a$.

Now, we will introduce the known results of favorite points, Erdős and Taylor [16] showed that for simple random walk in $d \geq 3$

$$\lim_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}^d} K(n, x)}{\log n} = \frac{1}{-\log P(T_0 < \infty)} \quad \text{a.s.}$$

For $d = 2$, they obtained

$$\frac{1}{4\pi} \leq \liminf_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}^2} K(n, x)}{(\log n)^2} \leq \limsup_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}^2} K(n, x)}{(\log n)^2} \leq \frac{1}{\pi} \quad \text{a.s.,}$$

and conjectured that the limit exists and equals $1/\pi$ a.s. Forty years later, Dembo et al. [8] verified this conjecture. In fact, they showed that for a simple random walk in \mathbb{Z}^2

$$\lim_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}^2} K(\tau_n, x)}{(\log n)^2} = \lim_{n \rightarrow \infty} \frac{4 \max_{x \in \mathbb{Z}^2} K(n, x)}{(\log n)^2} = \frac{4}{\pi} \quad \text{a.s.}$$

After this, [20] made another proof of this. As [10] mentioned, for $0 < \alpha < 1$ they defined the set of α -favorite points in \mathbb{Z}^2 such that

$$\Psi_n(\alpha) := \left\{ x \in D(0, n) : K(\tau_n, x) \geq \left\lceil \frac{4\alpha}{\pi} (\log n)^2 \right\rceil \right\}. \quad (3)$$

We clear up the geometric structure of $\Psi_n(\alpha)$ and attend the problem in [10] as follows.

Theorem 2.1. *For any $0 < \alpha, \beta < 1$ with probability one*

$$\lim_{n \rightarrow \infty} \frac{\log |\{(x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^\beta\}|}{\log n} = \rho_2(\alpha, \beta),$$

where

$$\rho_2(\alpha, \beta) := \begin{cases} 2 + 2\beta - \frac{4\alpha}{2-\beta} & (\beta \leq 2(1 - \sqrt{\alpha})), \\ 8(1 - \sqrt{\alpha}) - 4(1 - \sqrt{\alpha})^2/\beta & (\beta \geq 2(1 - \sqrt{\alpha})). \end{cases}$$

Theorem 2.2. *For any $0 < \alpha, \beta < 1$*

$$\lim_{n \rightarrow \infty} \frac{\log E[|\{(x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^\beta\}|]}{\log n} = \hat{\rho}_2(\alpha, \beta),$$

where

$$\hat{\rho}_2(\alpha, \beta) := \begin{cases} 2 + 2\beta - \frac{4\alpha}{2-\beta} & (\beta \leq 2 - \sqrt{2\alpha}), \\ 6 - 4\sqrt{2\alpha} & (\beta \geq 2 - \sqrt{2\alpha}). \end{cases}$$

To state the relation between $\hat{\rho}_2(\alpha, \beta)$ and $\rho_2(\alpha, \beta)$, for $\gamma > 0$ let

$$F_{h,\beta}(\gamma) := \gamma^2(1 - \beta) + \frac{h}{\beta}(1 - \gamma(1 - \beta))^2.$$

It can be easily checked that

$$\rho_2(\alpha, \beta) = 2 + 2\beta - 2\alpha \inf_{\gamma \in \Gamma_{\alpha,\beta}} F_{2,\beta}(\gamma),$$

where

$$\begin{aligned} \Gamma_{\alpha,\beta} &= \{\gamma \geq 0 : 2 - 2\beta - 2\alpha F_{0,\beta}(\gamma) \geq 0\} \\ &= \{\gamma \geq 0 : \alpha\gamma^2 \leq 1\}. \end{aligned}$$

It is also easy to verify that

$$\hat{\rho}_2(\alpha, \beta) = \sup_{\beta' \leq \beta} \sup_{\gamma \geq 0} \{2 + 2\beta' - 2\alpha F_{2,\beta'}(\gamma)\}, \quad (4)$$

so the difference between $\rho_2(\alpha, \beta)$ and $\hat{\rho}_2(\alpha, \beta)$ is the supremum in (4) is not subject to the constraint that $\gamma \in \Gamma_{\alpha,\beta}$. See Subsection 1.2 in [5] for more details of this difference.

Now, we introduce corresponding known results about special points of a GFF and a simple random walk in \mathbb{Z}_n^2 . First, we explain the high points of the GFF in \mathbb{Z}_n^2 . Originally, Bolthausen, Deuschel and Giacomin [2] showed that in probability

$$\lim_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}_n^2} \phi_n(x)}{\log n} = 2\sqrt{\frac{2}{\pi}},$$

where $\{\phi_n(x)\}_{x \in \mathbb{Z}_n^2}$ is the GFF whose definition equals is same that of [2, 4]. In what follows, for $0 < \alpha < 1$, we define the set of α -high points of the GFF such that

$$\mathcal{V}_n(\alpha) := \left\{ x \in \mathbb{Z}_n^2 : \frac{\phi_n(x)^2}{2} \geq \frac{4\alpha}{\pi}(\log n)^2 \right\}.$$

[4] showed if $\Psi_n(\alpha)$ in Theorems 2.1 and 2.2 are changed into $\mathcal{V}_n(\alpha)$, the exponent is same as Theorems 2.1 and 2.2. In addition, there are some interesting results concerning a relation between the local time of a random walk and a GFF. Eisenbaum, Kaspi, Marcus, Rosen, and Shi [13] gave a powerful equivalence in law called “generalized second Ray-Knight theorem”. Ding, Lee, and Peres [11, 12] gave a strong connection between the expected maximum of the GFF and the expected cover time.

Next, we explain late points of a simple random walk in \mathbb{Z}_n^2 . Originally, Dembo, Peres, Rosen and Zeitouni [9] showed that for a simple random walk in \mathbb{Z}_n^2 in probability

$$\lim_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}_n^2} T_x}{(n \log n)^2} = \frac{4}{\pi}.$$

(Now, we consider T_x as the stopping time to x of simple random walk in \mathbb{Z}_n^2 .) In what follows, for $0 < \alpha < 1$, we define the set of α -late points in \mathbb{Z}_n^2 such that

$$\mathcal{L}_n(\alpha) := \left\{ x \in \mathbb{Z}_n^2 : \frac{T_x}{(n \log n)^2} \geq \frac{4\alpha}{\pi} \right\}.$$

[3] obtained the result for $\mathcal{L}_n(\alpha)$ corresponding to Theorem 2.2, verifying the same formula but with $\mathcal{L}_n(\alpha)$ in place of $\Psi_n(\alpha)$. Similarly [10] obtained the one corresponding to Theorem 2.1.

In addition, as well as the conjecture in [10], Dembo [5, 6] suggested the following problem.

Open problem 1 (Open problem 4 in [5]). *In Theorem 1.4 of [10] the power growth exponent of pairs of α -late points within distance n^β of each other is computed. Extend this to a “full multi-fractal analysis”.*

We have solved this problem. In fact, we estimate the following $\rho_j(\alpha, \beta)$ for any $0 < \alpha, \beta < 1$, $j \geq 3$:

$$\lim_{n \rightarrow \infty} \frac{\log |\{(x_1, x_2, \dots, x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \leq n^\beta \text{ for any } 1 \leq i, l \leq j\}|}{\log n} = \rho_j(\alpha, \beta).$$

In addition, we have extended this problem to $\Psi_n(\alpha)$, $\mathcal{V}_n(\alpha)$. We can extend the condition “ $d(x_i, x_l) \leq n^\beta$ ” to more general ones. We will discuss all these results in the forthcoming papers.

In the proofs given in the rest of the paper, we use the letters C or c for constants that may vary from place to place.

3 Basic properties

In this section, we introduce a few estimates as elementary properties of simple random walk in \mathbb{Z}^2 and finite Markov chains for later use.

3.1 Hitting probabilities

First, we compute probabilities that simple random walk in \mathbb{Z}^2 doesn't hit two points until a certain random time. Given two distinct points x_1 and x_2 of \mathbb{Z}^2 and a non-empty subset of \mathbb{Z}^2 disjoint of $\{x_1, x_2\}$, let $\tilde{\tau}$ denote the first hitting time when the simple random walk enter this set and define for $i, l \in \{1, 2\}$,

$$W_{i,l} := \sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < \tilde{\tau}).$$

By decomposing the probability $1 = P^{x_i}(\tilde{\tau} < \infty)$ by means of the last time the walk leaves the set $\{x_1, x_2\}$ before $\tilde{\tau}$ we obtain

$$1 = W_{i,1}P^{x_1}(\tilde{\tau} < T_{x_1,x_2}) + W_{i,2}P^{x_2}(\tilde{\tau} < T_{x_1,x_2}),$$

for $i \in \{1, 2\}$. Since $W_{1,2} = P^{x_1}(T_{x_2} < \tilde{\tau})W_{2,2} < W_{2,2}$ and similarly $W_{2,1} < W_{1,1}$, we have

$$W_{1,1}W_{2,2} - W_{1,2}W_{2,1} \neq 0,$$

and solving the above linear system yields

$$\begin{aligned} P^{x_1}(\tilde{\tau} < T_{x_1,x_2}) &= \frac{W_{2,2} - W_{1,2}}{W_{1,1}W_{2,2} - W_{1,2}W_{2,1}}, \\ P^{x_2}(\tilde{\tau} < T_{x_1,x_2}) &= \frac{W_{1,1} - W_{2,1}}{W_{1,1}W_{2,2} - W_{1,2}W_{2,1}}. \end{aligned} \quad (5)$$

Next, we introduce estimates of the hitting probability for simple random walk in \mathbb{Z}^2 . By Exercise 1.6.8 of [17] or (4.1) and (4.3) in [20], we have that, in $0 < r < |x| < R$,

$$\begin{aligned} P^x(T_0 < \tau_R) &= \frac{\log(R/|x|) + O(|x|^{-1})}{\log R} (1 + O((\log |x|)^{-1})), \\ P^x(\tau_r < \tau_R) &= \frac{\log(R/|x|) + O(r^{-1})}{\log(R/r)}. \end{aligned} \quad (6)$$

In addition, by Proposition 1.6.7 in [17] or (2.2) in [20], we have that for any $x \in D(0, n)$,

$$\sum_{m=0}^{\infty} P^x(S_m = 0, m < \tau_n) = \begin{cases} \frac{2}{\pi} \log \left(\frac{n}{|x|} \right) + O(|x|^{-1} + n^{-1}) & \text{for } x \neq 0, \\ \frac{2}{\pi} \log n + O(1) & \text{for } x = 0. \end{cases} \quad (7)$$

Therefore, the strong Markov property yields

$$P(\tau_n < T_0) = \left(\sum_{m=0}^{\infty} P(S_m = 0, m < \tau_n) \right)^{-1} = \frac{\pi}{2 \log n} (1 + o(1)). \quad (8)$$

3.2 Probabilities conditioned by the local time

Now, we prepare some expression of the probability of events associated with a Markov chain on $\{1, 2, 3\}$. It will be used when considering transitions of the random walk in \mathbb{Z}^2 from a given set to another one. We can reduce such a problem to that for Markov chain by considering a sequence of hitting times. Consider the general Markov chain $\{X_m\}_{m=0}^{\infty}$ of discrete time on the state space $\{1, 2, 3\}$. Let $d_{i,l}$ be the transition probability of $\{X_m\}_{m=0}^{\infty}$ and $\mathbb{N} := \{1, 2, \dots\}$. We claim that for any $n_1, n_2 \in \mathbb{N}$,

$$\begin{aligned} P^1 \left(\sum_{m=1}^{n_1+n_2} 1_{\{X_m=1\}} = n_1, \sum_{m=1}^{n_1+n_2} 1_{\{X_m=2\}} = n_2, X_{n_1+n_2} = 2 \right) \\ = \sum_{0 \leq i \leq n_1 \wedge (n_2-1)} d_{1,1}^{n_1-i} d_{1,2}^{i+1} \frac{n_1!}{(n_1-i)!i!} d_{2,1}^i d_{2,2}^{n_2-i-1} \frac{(n_2-1)!}{(n_2-i-1)!i!} \end{aligned} \quad (9)$$

and

$$\begin{aligned}
& P^1 \left(\sum_{m=1}^{n_1+n_2} 1_{\{X_m=1\}} = n_1, \sum_{m=1}^{n_1+n_2} 1_{\{X_m=2\}} = n_2, X_{n_1+n_2} = 1 \right) \\
&= \sum_{1 \leq i \leq n_1 \wedge n_2} d_{1,1}^{n_1-i} d_{1,2}^i \frac{n_1!}{(n_1-i)!i!} d_{2,1}^i d_{2,2}^{n_2-i} \frac{(n_2-1)!}{(n_2-i)!(i-1)!}, \tag{10}
\end{aligned}$$

where $0! = 1$. We will show (9) and (10) by induction on $n_1 + n_2$. It is trivial that (9) and (10) hold for $n_1 + n_2 = 2$, that is, $n_1 = n_2 = 1$. Fix $n \geq 2$ and assume that (9) and (10) hold for any $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 = n$. First, we will show that (9) holds for any $\tilde{n}_1, \tilde{n}_2 \in \mathbb{N}$ with $\tilde{n}_1 + \tilde{n}_2 = n + 1$. When $\tilde{n}_2 = 1$,

$$\begin{aligned}
& P^1 \left(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=1\}} = \tilde{n}_1, \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=2\}} = \tilde{n}_2, X_{\tilde{n}_1+\tilde{n}_2} = 2 \right) \\
&= P^1 \left(X_m = 1 \text{ for } 1 \leq m \leq \tilde{n}_1 + \tilde{n}_2 - 1, X_{\tilde{n}_1+\tilde{n}_2} = 2 \right) \\
&= d_{1,1}^{\tilde{n}_1+\tilde{n}_2-1} d_{1,2},
\end{aligned}$$

and, hence, the claim holds. In addition, the Markov property yields that for any $\tilde{n}_2 \geq 2$ and $\tilde{n}_1 \in \mathbb{N}$,

$$\begin{aligned}
& P^1 \left(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=1\}} = \tilde{n}_1, \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=2\}} = \tilde{n}_2, X_{\tilde{n}_1+\tilde{n}_2} = 2 \right) \\
&= d_{1,1} P^1 \left(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=1\}} = \tilde{n}_1 - 1, \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=2\}} = \tilde{n}_2, X_{\tilde{n}_1+\tilde{n}_2-1} = 2 \right) \\
&\quad + d_{1,2} P^2 \left(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=1\}} = \tilde{n}_1, \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=2\}} = \tilde{n}_2 - 1, X_{\tilde{n}_1+\tilde{n}_2-1} = 2 \right) \\
&= d_{1,1} \sum_{0 \leq i \leq (\tilde{n}_1-1) \wedge (\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-1-i} d_{1,2}^{i+1} \frac{(\tilde{n}_1-1)!}{(\tilde{n}_1-1-i)!i!} d_{2,1}^i d_{2,2}^{\tilde{n}_2-1-i} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-1-i)!i!} \\
&\quad + d_{1,2} \sum_{1 \leq i \leq \tilde{n}_1 \wedge (\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-i} d_{1,2}^i \frac{(\tilde{n}_1-1)!}{(\tilde{n}_1-i)!(i-1)!} d_{2,1}^i d_{2,2}^{\tilde{n}_2-1-i} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-1-i)!i!} \\
&= \sum_{0 \leq i \leq \tilde{n}_1 \wedge (\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-i} d_{1,2}^{i+1} \frac{\tilde{n}_1!}{(\tilde{n}_1-i)!i!} d_{2,1}^i d_{2,2}^{\tilde{n}_2-i-1} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-i-1)!i!}.
\end{aligned}$$

The second equality comes from the assumption and the symmetry of sites 1 and 2. Therefore, we obtain (9). In addition, the same argument yields the corresponding result for (10), and, hence, we obtain the desired result.

4 Proof of Theorem 2.2

4.1 Proof of the lower bound in Theorem 2.2

To show Theorem 2.2, we prepare some notations to make a correspondence of the framework of Subsection 3.2 to that of the present one. Let $\tilde{\alpha} := \lceil \frac{4\alpha}{\pi} (\log n)^2 \rceil$, for x, x' be two distinct points of $D(0, n)$, and set

$$(U_1, U_2, U_3) := (x, x', \partial D(0, n))$$

and

$$b_{i,l} := P^{U_i}(\min_{s \in \{1,2,3\}} T_{U_s} = T_{U_l}),$$

for $i \in \{1, 2\}$ and $l \in \{1, 2, 3\}$. In addition, for $p \geq 2$ and $x \in \mathbb{Z}^2$, let

$$T_x^1 := \inf\{m \geq 0 : S_m = x\}, \quad T_x^p := \inf\{m > T_x^{p-1} : S_m = x\}.$$

As a first step, we will estimate $P(x, x' \in \Psi_n(\alpha))$. By the symmetry of x and x' , we may assume that $b_{1,1} \leq b_{2,2}$ and

$$P(T_{x'} < T_x \wedge \tau_n) \leq P(T_x < T_{x'} \wedge \tau_n) \quad (11)$$

all the other three cases can be discussed in the same way as we will see below. Note that the time-reversal of simple random walk yields that $b_{1,2} = b_{2,1}$. Then, (9) yields

$$\begin{aligned} P(x, x' \in \Psi_n(\alpha)) &\geq P(T_x < T_{x'} \wedge \tau_n) P^x(x, x' \in \Psi_n(\alpha), T_x^{\tilde{\alpha}} > T_{x'}^{\tilde{\alpha}}) \\ &\geq P(T_x < T_{x'} \wedge \tau_n) \\ &\quad \times \sum_{0 \leq i \leq \tilde{\alpha}-1} b_{1,1}^{\tilde{\alpha}-1-i} b_{1,2}^{i+1} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} b_{2,1}^i b_{2,2}^{\tilde{\alpha}-1-i} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} \\ &= P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \\ &\quad \times \sum_{0 \leq i \leq \tilde{\alpha}-1} b_{1,1}^{\tilde{\alpha}-1-i} b_{1,2}^i \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} b_{2,1}^i b_{2,2}^{\tilde{\alpha}-1-i} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} \\ &\geq P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \left(\max_{0 \leq i \leq \tilde{\alpha}-1} b_{1,1}^{\tilde{\alpha}-1-i} b_{1,2}^i \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} \right)^2 \\ &\geq P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \frac{1}{\tilde{\alpha}^2} (b_{1,1} + b_{1,2})^{2\tilde{\alpha}-2} \\ &= P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \frac{1}{\tilde{\alpha}^2} P^x(T_{x,x'} < \tau_n)^{2\tilde{\alpha}-2}. \end{aligned} \quad (12)$$

The right most number of (12), we estimate each of probabilities involved in it. Set $s := \log d(x, x') / \log n$ and consider $0 < \epsilon < 1$. Then, by (7), we obtain that for sufficiently large $n \in \mathbb{N}$, uniformly in $x, x' \in D(0, n^{1-\epsilon})$,

$$\sum_{m=0}^{\infty} P^{U_i}(S_m \in U_l, m < \tau_n) = \begin{cases} \frac{2}{\pi}(1+o(1)) \log n & \text{if } i = l, \\ \frac{2(1-s)}{\pi}(1+o(1)) \log n & \text{if } i \neq l, \end{cases} \quad (13)$$

and, hence, if we set $\tilde{\tau} = \tau_n$ in (5), we have

$$b_{1,3} = \frac{\pi(1+o(1))}{2(2-s) \log n}. \quad (14)$$

Then, we obtain that, uniformly in x, x' ,

$$\begin{aligned} P^x(T_{x,x'} < \tau_n)^{2\tilde{\alpha}} &= \exp(-2\tilde{\alpha}b_{1,3} + o(1) \log n) \\ &= \exp\left(-\frac{\tilde{\alpha}\pi}{(2-s) \log n} + o(1) \log n\right). \end{aligned} \quad (15)$$

In addition, (8) yields that, uniformly in x ,

$$P^x(\tau_n < T_x) = \frac{\pi}{2 \log n} (1 + o(1)) \quad (16)$$

and (6) yields that for any $\epsilon > 0$ there exists $c > 0$ such that for all sufficiently large $n \in \mathbb{N}$

$$P^x(T_{x'} < \tau_n) \geq P^x(T_{x'} < T_{\partial D(x', n/2)}) \geq \frac{\log(n/2n^{1-\epsilon})}{\log n/2} (1 + O((\log n)^{-1})) \geq c, \quad (17)$$

$$P(T_x < \tau_n) = P^x(T_0 < \tau_n) \geq \frac{\log(n/n^{1-\epsilon})}{\log n} (1 + O((\log n)^{-1})) \geq c, \quad (18)$$

where the equality comes from the time-reversal of simple random walk. Note that

$$P^x(T_{x'} < \tau_n) = \sum_{i=0}^{\infty} b_{1,1}^i b_{1,2} = \frac{1}{1 - b_{1,1}} b_{1,2}$$

and, hence, by (16) and (17), for all sufficiently large $n \in \mathbb{N}$,

$$b_{1,2} \geq P^x(\tau_n < T_x) P^x(T_{x'} < \tau_n) \geq \frac{c}{\log n}.$$

In addition, (11) and (18) yield that there exists $c > 0$ such that for any $n \in \mathbb{N}$,

$$P(T_x < T_{x'} \wedge \tau_n) \geq \frac{1}{2} P(T_{x,x'} < \tau_n) \geq \frac{1}{2} P(T_x < \tau_n) \geq c.$$

We now go back to (12). By applying all these estimates, for any $\delta > 0$ and $\epsilon > 0$ there exists such that $n_0 \in \mathbb{N}$ for $n \geq n_0$ and $x, x' \in D(0, n^{1-\epsilon})$,

$$P(x, x' \in \Psi_n(\alpha)) \geq \exp\left(-\frac{\tilde{\alpha}\pi}{(2-s)\log n} - \delta \log n\right). \quad (19)$$

Now we turn to the desired lower bound. By (19), for any $\delta > 0$, $\epsilon > 0$, $\beta' \in [\epsilon, \beta]$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} E[\{ (x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^\beta \}] &\geq \sum_{\substack{x, x' \in D(0, n^{1-\epsilon}), \\ 0 < d(x, x') \leq n^{\beta'}}} P(x, x' \in \Psi_n(\alpha)) \\ &\geq \sum_{\substack{x, x' \in D(0, n^{1-\epsilon}), \\ 0 < d(x, x') \leq n^{\beta'}}} \exp\left(-\frac{\tilde{\alpha}\pi}{(2-\beta')\log n} - \delta \log n\right) \\ &\geq n^{2+2\beta'-2\epsilon-4\alpha/(2-\beta')-2\delta}. \end{aligned}$$

Note that $2 + 2\beta' - 4\alpha/(2 - \beta')|_{\beta'=\beta \wedge (2-\sqrt{2\alpha})} = \hat{\rho}_2(\alpha, \beta)$. Then, since $\delta > 0$ and $\epsilon > 0$ are arbitrary, we obtain the desired result.

4.2 Proof of the upper bound in Theorem 2.2

The idea of the upper bound in Theorem 2.2 is similar to that of the lower bound, while we do not require a combinatorial argument in Subsection 3.2. Indeed, the strong Markov property yields that

$$\begin{aligned}
& P(x, x' \in \Psi_n(\alpha)) \\
& \leq P(|\{l < \tau_n : S_l \in \{x', x\}\}| \geq 2\tilde{\alpha}) \\
& = P(T_x < T_{x'} \wedge \tau_n) P^x(|\{l < \tau_n : S_l \in \{x', x\}\}| \geq 2\tilde{\alpha} - 1) \\
& \quad + P(T_{x'} < T_x \wedge \tau_n) P^{x'}(|\{l < \tau_n : S_l \in \{x', x\}\}| \geq 2\tilde{\alpha} - 1) \\
& \leq \max_{y \in \{x, x'\}} P^y(|\{l < \tau_n : S_l \in \{x', x\}\}| \geq 2\tilde{\alpha} - 1) \\
& \leq \max_{y' \in \{x, x'\}} P^{y'}(T_{x, x'} < \tau_n) \max_{y \in \{x, x'\}} P^y(|\{l < \tau_n : S_l \in \{x', x\}\}| \geq 2\tilde{\alpha} - 2) \\
& \leq \dots \\
& \leq \max_{y' \in \{x, x'\}} P^{y'}(T_{x, x'} < \tau_n)^{2\tilde{\alpha}-1} \\
& \leq \max_{y' \in \{x, x'\}} P^{y'}(T_{x, x'} < \tau_{2n})^{2\tilde{\alpha}-1}.
\end{aligned} \tag{20}$$

Then, letting $\tilde{\tau} = \tau_{2n}$ in (5), and arguing similarly as we obtained (13) and (14), we obtain

$$1 - \max_{y' \in \{x, x'\}} P^{y'}(T_{x, x'} < \tau_{2n}) = \frac{\pi(1 + o(1))}{2(2 - s) \log n},$$

uniformly in $x, x' \in D(0, n)$. Hence, by (20), for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $x, x' \in D(0, n)$,

$$P(x, x' \in \Psi_n(\alpha)) \leq \exp \left(- \frac{\tilde{\alpha}\pi}{(2 - s) \log n} + \delta \log n \right).$$

Unlike the lower bound, we need more estimate to bound the summation of $P(x, x' \in \Psi_n(\alpha))$ in (x, x') . Since $\min\{1/\sqrt{\alpha}, 2/(2 - \beta)\} \geq 1$,

$$\hat{\rho}_2(\alpha, \beta) \geq \rho_2(\alpha, \beta) \geq 2 + 2\beta - 2\alpha F_{2, \beta}(1) = 2(1 - \alpha) + 2\beta(1 - \alpha). \tag{21}$$

In addition, (8) yields that for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$\begin{aligned}
E[|\Psi_n(\alpha)|] &= \sum_{x \in D(0, n)} P(x \in \Psi_n(\alpha)) \\
&\leq \sum_{x \in D(0, n)} P^x(T_x < \tau_n)^{\tilde{\alpha}-1} \\
&\leq \sum_{x \in D(0, n)} P^x(T_x < T_{\partial D(x, 2n)})^{\tilde{\alpha}-1} \\
&\leq 4n^2 \times \left(1 - \frac{\pi}{2 \log n} + o\left(\frac{1}{\log n}\right) \right)^{\tilde{\alpha}-1} \leq n^{2(1-\alpha)+\delta}.
\end{aligned} \tag{22}$$

Now we are ready to enter the main part of the proof. By (22), we obtain

$$\begin{aligned}
& E[|\{(x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^{\beta(1-\alpha)}\}|] \\
& \leq E[|\Psi_n(\alpha)|] \times C n^{2\beta(1-\alpha)} \leq n^{\hat{\rho}_2(\alpha, \beta)+\delta}.
\end{aligned} \tag{23}$$

In addition, for any $\delta > 0$, $0 < h < 1$, $\beta' \in [\beta(1 - \alpha)h^{-1}, \beta]$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}
& E[|\{(x, x') \in \Psi_n(\alpha)^2 : n^{\beta'h} < d(x, x') \leq n^{\beta'}\}|] \\
&= \sum_{\substack{x, x' \in D(0, n), \\ n^{\beta'h} < d(x, x') \leq n^{\beta'}}} P(x, x' \in \Psi_n(\alpha)) \\
&\leq \sum_{\substack{x, x' \in D(0, n), \\ n^{\beta'h} < d(x, x') \leq n^{\beta'}}} \exp\left(-\frac{\tilde{\alpha}\pi}{(2 - \beta'h)\log n} + 2\delta \log n\right) \\
&\leq n^{2+2\beta'-4\alpha/(2-\beta'h)+3\delta}.
\end{aligned} \tag{24}$$

Note that $\max_{(1-\alpha)\beta \leq \beta' \leq \beta} 2 + 2\beta' - 4\alpha/(2 - \beta') = \hat{\rho}_2(\alpha, \beta)$. Then, since $\delta > 0$ and $0 < h < 1$ are arbitrary, (23) and (24) yield the desired result.

5 Key lemma for the proof of the upper bound in Theorem 2.1

This section provides large deviation estimates that are keys to the proof of the upper bound in Theorem 2.1. In addition, we start by defining the several sequence. Let $r_0 := 0$ and $r_k := (k!)^3$ for $k \in \mathbb{N}$. Set $K_n := n^3 r_n$ for $n \in \mathbb{N}$. Now fix $0 < \alpha, \beta < 1$. Set $n_k = 6\alpha(n - k)^2 \log k$. For $z \in D(0, K_n)$, let $N_{n,k}^z$, denote the number of excursions from $\partial D(z, r_k)$ to $\partial D(z, r_{k-1})$ until time τ_{K_n} . To state the assertion, we define the domain \mathbf{Go}^β and $\mathbf{Go}^{h,\beta}$ by

$$\begin{aligned}
\mathbf{Go}^\beta &:= \{(z, x) : z \in D(0, K_n), x \in D(z, r_{\beta n-2}) \cap D(0, K_n)\}, \\
\mathbf{Go}^{h,\beta} &:= \{(z, x, x') : (z, x), (z, x') \in \mathbf{Go}^\beta, d(x, x') \geq r_{\beta h n/2-3}\}.
\end{aligned}$$

Let $\tilde{\Psi}_n(\alpha) := \{x : K(\tau_n, x) \in [4\alpha/\pi(\log n)^2, 4/\pi(\log n)^2]\}$.

Lemma 5.1.

(1) For any $\delta > 0$, there exists $C > 0$ such that for any $\gamma \geq 0$,

$$\max_{z \in \mathbb{Z}^2} P\left(\frac{N_{n,\beta n}^z}{n_{\beta n}} \geq \gamma^2\right) \leq C K_n^{-2\alpha F_{0,\beta}(\gamma) + \delta}. \tag{25}$$

(2) For any $\delta, \delta' > 0$, there exist $C > 0$ and $0 < h < 2$ (with h close to 2) such that for any $\gamma \in [0, 2/(2 - \beta)]$,

$$\max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \gamma^2 \leq \frac{N_{n,\beta n}^z}{n_{\beta n}} < \gamma^2 + \delta'\right) \leq C K_n^{-2\alpha F_{2,\beta}(\gamma) + \delta}. \tag{26}$$

From (26), we obtain the following bound, which we will use in the sequel in a crucial way.

Corollary 5.1. For any $\delta > 0$, there exist $C, \delta_0 > 0$ such that

$$\max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \frac{N_{n,\beta n}^z}{n_{\beta n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right) \leq C K_n^{-2\alpha F_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\}) + 2\delta}.$$

Proof. Take $\delta_0 > 0$ satisfying

$$2\alpha|F_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\}) + \delta_0 - F_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\})| \leq \delta. \tag{27}$$

If we set $\delta' = \delta_0$ and $C' = \lceil ((1/\sqrt{\alpha} + \delta_0)^2 + \delta_0)/\delta_0 \rceil$, (26) yields that

$$\begin{aligned}
& \max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \frac{N_{n,\beta n}^z}{n_{\beta n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right) \\
& \leq C' \max_{\gamma' \leq (1/\sqrt{\alpha} + \delta_0)^2} K_n^{-2\alpha F_{2,\beta}(\sqrt{\gamma'}) + \delta} \\
& = C' K_n^{-2\alpha F_{2,\beta}(\min\{1/\sqrt{\alpha} + \delta_0, 2/(2-\beta)\}) + \delta} \\
& \leq C' K_n^{-2\alpha F_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\}) + 2\delta}.
\end{aligned}$$

Here the equality comes from the fact that $F_{2,\beta}$ is minimized at $2/(2-\beta)$ and the last inequality is a consequence of the choice of δ_0 in (27). Hence, we obtain the desired result. \square

Let us enter the proof of Lemma 5.1. The proof of the first assertion (25) is rather simple.

Proof of Lemma 5.1 (1). Substituting $R = K_n$ and $r = r_{\beta n-1}$ for (6), the strong Markov property provides

$$\begin{aligned}
\max_{z \in D(0, K_n)} P\left(\frac{N_{n,\beta n}^z}{n_{\beta n}} \geq \gamma^2\right) & \leq \max_{z \in D(0, K_n), y \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < T_{\partial D(z, 2K_n)})^{\gamma^2 n_{\beta n}} \\
& \leq \max_{y \in \partial D(0, r_{\beta n})} P^y(\tau_{r_{\beta n-1}} < \tau_{2K_n})^{\gamma^2 n_{\beta n}} \\
& \leq \left(\frac{\log(2K_n/r_{\beta n}) + O(r_{\beta n-1}^{-1})}{\log(2K_n/r_{\beta n-1})}\right)^{\gamma^2 n_{\beta n}} \\
& = \left(1 - \frac{1 + o(1)}{n - \beta n}\right)^{\gamma^2 n_{\beta n}} \\
& = \exp(-6\alpha\gamma^2(n - \beta n) \log n + o(n \log n)) \\
& = C K_n^{-2\alpha F_{0,\beta}(\gamma) + o(1)}
\end{aligned}$$

and, hence, the desired result holds. \square

To show the second assertion (26), we define and estimate the following probabilities. For any $(z, x, x') \in \mathbf{Go}^{h,\beta}$, let

$$(V_1, V_2) := (\partial D(z, r_{\beta n}), \{x, x'\})$$

and for $l = 1, 2$,

$$\begin{aligned}
a_{1,l} &:= \max_{y \in \partial D(z, r_{\beta n-1})} P^y\left(\min_{s \in \{1,2\}} T_{V_s} = T_{V_l}\right), \\
a_{2,l} &:= \max_{y \in \{x, x'\}} P^y\left(\min_{s \in \{1,2\}} T_{V_s} = T_{V_l}\right).
\end{aligned}$$

Lemma 5.2. *For any $\epsilon > 0$ there exists $0 < h < 2$ such that for any $(z, x, x') \in \mathbf{Go}^{h,\beta}$ and all sufficiently large $n \in \mathbb{N}$, the following hold:*

$$a_{1,1} \leq 1 - \frac{2 - \epsilon}{\beta n}, \quad a_{1,2} \leq \frac{2 + \epsilon}{\beta n}, \quad (28)$$

$$a_{2,1} \leq \frac{\pi + \epsilon}{6\beta n \log n}, \quad a_{2,2} \leq 1 - \frac{\pi - \epsilon}{6\beta n \log n}. \quad (29)$$

Proof. Fix $\epsilon > 0$. First, we will show (28). By (6.16) in [10], we obtain for all sufficiently large $n \in \mathbb{N}$,

$$a_{1,1} \leq 1 - \frac{2 - \epsilon}{\beta n}.$$

In addition, since $D(x, r_{\beta n-1}/2) \subset D(z, r_{\beta n-1})$ and $D(z, r_{\beta n}) \subset D(x, 2r_{\beta n})$ hold for $(z, x) \in \mathbf{Go}^\beta$, (6) yields that for $y' \in \{x, x'\}$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \max_{y' \in \partial D(z, r_{\beta n-1})} P^y(T_{y'} < T_{\partial D(z, r_{\beta n})}) &\leq \max_{y' \in D(y', r_{\beta n-1}/2)^c} P^y(T_{y'} < T_{\partial D(y', 2r_{\beta n})}) \\ &= \max_{y' \in \partial D(y', r_{\beta n-1}/2)} P^y(T_{y'} < T_{\partial D(y', 2r_{\beta n})}) \\ &\leq \frac{\log(4r_{\beta n}/r_{\beta n-1}) + O(r_{\beta n-1}^{-1})}{\log r_{\beta n}} (1 + o(1)) \\ &\leq \frac{1 + \epsilon}{\beta n}. \end{aligned}$$

Since $a_{1,2} \leq \max_{y' \in \partial D(z, r_{\beta n-1})} P^y(T_x < T_{\partial D(z, r_{\beta n})}) + \max_{y' \in \partial D(z, r_{\beta n-1})} P^y(T_{x'} < T_{\partial D(z, r_{\beta n})})$, we have (28). Next, we will estimate (29). By (7) and (8), for any $(z, x, x') \in \mathbf{Go}^{h,\beta}$ and $y, y' \in \{x, x'\}$,

$$\sum_{m=0}^{\infty} P^y(S_m = y', m < T_{\partial D(z, r_{\beta n})}) \begin{cases} = \frac{2+o(1)}{\pi} 3\beta n \log n & \text{if } y = y', \\ \leq \frac{2+o(1)}{\pi} \frac{3\beta(2-h)}{2} n \log n & \text{if } y \neq y'. \end{cases} \quad (30)$$

Hence, if we substitute $T_{\partial D(z, r_{\beta n})}$ into $\tilde{\tau}$ in (5), (30) and picking $h < 2$ close to 2 yield (29) for all sufficiently large $n \in \mathbb{N}$. \square

Proof of Lemma 5.1 (2). Let

$$\tilde{T} := \inf\{m > T_{\partial D(z, r_{\beta n-1})} : S_m \in \partial D(z, r_{\beta n})\}$$

and

$$\begin{aligned} a'_{1,1} &:= \max_{y' \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}, \tilde{T} < T_{x_1, x_2}), \\ a'_{1,2} &:= \max_{y' \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}, \tilde{T} > T_{x_1, x_2}). \end{aligned}$$

Then, the strong Markov property yields for any $c_1, c_2, c_3 \in \mathbb{N}$,

$$\begin{aligned} &\max_{(z, x, x') \in \mathbf{Go}^{h,\beta}} P(K(\tau_{K_n}, x) = c_2, K(\tau_{K_n}, x') = c_3, N_{n, \beta n}^z = c_1) \\ &\leq \max_{(z, x, x') \in \mathbf{Go}^{h,\beta}} \max_{y \in \{x, x'\}} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N_{n, \beta n}^z = c_1) \\ &+ \max_{\substack{(z, x, x') \in \mathbf{Go}^{h,\beta}, \\ y \in \partial D(z, r_{\beta n})}} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N_{n, \beta n}^z = c_1). \end{aligned} \quad (31)$$

Now we use weight $a_{2,l}$ (or $a'_{1,l}$) instead of $d_{i,l}$ in (9) or (10). For $z \in D(0, K_n)$, let $M_{n,k}^z$ be the number of excursions from $\partial D(z, r_{k-1})$ to $\partial D(z, r_k)$ until time τ_{K_n} . Note that if we consider the simple random walk starting at x or x' , it holds that

$$K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N_{n, \beta n}^z = c_1 \subset \{M_{n, \beta n}^z = c_1 + 1\},$$

then we can consider the path corresponding to (9) as $n_1 = c_2 + c_3 + 1$ and $n_2 = c_1 + 1$. In addition, if we consider the simple random walk starting at $\partial D(z, r_{\beta n})$, it holds that

$$K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N_{n, \beta n}^z = c_1 \subset \{M_{n, \beta n}^z = c_1\},$$

then we can consider the path corresponding to (9) as $n_1 = c_2 + c_3$ and $n_2 = c_1$. Therefore, for $c_1 \leq c_2 + c_3 - 1$, (9) and (10) yield

$$\begin{aligned} & \max_{y \in \{x, x'\}} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N_{n, \beta n}^z = c_1) \\ & \leq \sum_{0 \leq i \leq c_1} a'_{1,1}{}^{c_1-i} a'_{1,2}{}^i \frac{c_1!}{(c_1-i)!i!} a_{2,1}^{i+1} a_{2,2}^{c_2+c_3-1-i} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!i!}, \\ & \max_{y \in \partial D(z, r_{\beta n})} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N_{n, \beta n}^z = c_1) \\ & \leq \sum_{1 \leq i \leq c_1} a'_{1,1}{}^{c_1-i} a'_{1,2}{}^i \frac{c_1!}{(c_1-i)!i!} a_{2,1}^i a_{2,2}^{c_2+c_3-i} \frac{(c_2+c_3-1)!}{(c_2+c_3-i)!(i-1)!}. \end{aligned}$$

Let us denote the probability in the left hand side of (31) by A . Then the above estimate implies

$$A \leq 2(c_1 + 1) \max_{0 \leq i \leq c_1} a'_{1,1}{}^{c_1-i} a'_{1,2}{}^i \frac{c_1!}{(c_1-i)!i!} a_{2,1}^i a_{2,2}^{c_2+c_3-1-i} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!(i-1)!}. \quad (32)$$

Now, the strong Markov property yields that for any $i = 1, 2$

$$a'_{1,i} \leq \max_{y \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}) a_{1,i}.$$

Note that for any $0 \leq i \leq c_1$,

$$a'_{1,1}{}^{c_1-i} a'_{1,2}{}^i \leq \left(\max_{y \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}) \right)^{c_1} a_{1,1}^{c_1-i} a_{1,2}^i. \quad (33)$$

Now, we combine the last estimate with our assertion. For $\gamma \in [0, 2/(2-\beta)]$ and $\delta > 0$, we take c_1, c_2 and c_3 so that $\gamma^2 \leq c_1/n_{\beta n} < \gamma^2 + \delta_0$, $c_2, c_3 \in [4\alpha(\log K_n)^2/\pi, 4(\log K_n)^2/\pi]$. It holds that $c_1 \leq c_2 + c_3 - 1$ for sufficiently large $n \in \mathbb{N}$. Then, in a similar way as in (25), (6) yields that there exists $C > 0$ such that for all sufficiently large $n \in \mathbb{N}$,

$$\left(\max_{y \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}) \right)^{c_1} \leq CK_n^{-2\alpha F_{0,\beta}(\gamma) + \delta/4}. \quad (34)$$

Therefore, by applying (33) and (34) to (32), we obtain

$$A \leq CK_n^{-2\alpha F_{0,\beta}(\gamma) + \delta/4} \max_{0 \leq i \leq c_1} a_{1,1}^{c_1-i} a_{1,2}^i \frac{c_1!}{(c_1-i)!i!} a_{2,1}^i a_{2,2}^{c_2+c_3-1-i} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!(i-1)!}. \quad (35)$$

With the aid of Lemma 5.2 we will estimate the maximal term in (35). Let us define $g(i)$ by

$$\begin{aligned} g(i) &:= \left(1 - \frac{2-\epsilon}{\beta n}\right)^{c_1-i} \left(\frac{2+\epsilon}{\beta n}\right)^i \frac{c_1!}{(c_1-i)!i!} \\ &\quad \times \left(\frac{\pi+\epsilon}{6\beta n \log n}\right)^i \left(1 - \frac{\pi-\epsilon}{6\beta n \log n}\right)^{c_2+c_3-1-i} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!(i-1)!}. \end{aligned}$$

Then, by Lemma 5.2 and (35), we have $A \leq CK_n^{-2\alpha F_{0,\beta}(\gamma) + \delta/4} \max_{0 \leq i \leq c_1} g(i)$. Note that for any $1 \leq i \leq c_1$,

$$\frac{g(i-1)}{g(i)} = \frac{(1 - \frac{\pi-\epsilon}{6\beta n \log n})(1 - \frac{2-\epsilon}{\beta n})i(i-1)}{(\frac{\pi+\epsilon}{6\beta n \log n})(\frac{2+\epsilon}{\beta n})(c_2+c_3-i)(c_1+1-i)}.$$

By taking the growth order of c_1, c_2 and c_3 into account, the maximum of $g(i)$ among $0 \leq i \leq c_1$ is attained at

$$i = \left\lceil (1 + o(1)) \sqrt{(c_2 + c_3)c_1 \frac{2+\epsilon}{\beta n} \times \frac{\pi+\epsilon}{6\beta n \log n}} \right\rceil - 1.$$

In addition, the Stirling formula yields

$$\begin{aligned}
& g\left(\left[(1+o(1))\sqrt{(c_2+c_3)c_1\frac{2+\epsilon}{\beta n}\times\frac{\pi+\epsilon}{6\beta n\log n}}\right]-1\right) \\
& \leq K_n^{-4\alpha(\sqrt{(c'_2+c'_3)/2}-\gamma(1-\beta))^2/\beta+f(\epsilon)+\delta/8} \\
& \leq K_n^{-4\alpha(1-\gamma(1-\beta))^2/\beta+f(\epsilon)+\delta/8},
\end{aligned}$$

where $f(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, $c'_2 = c_2\pi/(4\alpha(\log K_n)^2)$ and $c'_3 = c_3\pi/(4\alpha(\log K_n)^2)$. The last inequality comes from $\gamma \leq 2/(2-\beta) \leq 1/(1-\beta)$ for $\gamma \in [0, 2/(2-\beta)]$. Hence, we obtain

$$\begin{aligned}
& \max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P(K(\tau_{K_n}, x) = c_2, K(\tau_{K_n}, x') = c_3, N_{n,\beta n}^z = c_1) \\
& \leq CK_n^{-2\alpha F_{0,\beta}(\gamma)+\delta/4} K_n^{-4\alpha(1-\gamma(1-\beta))^2/\beta+\delta/4} \\
& = CK_n^{-2\alpha F_{2,\beta}(\gamma)+\delta/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \gamma^2 \leq \frac{N_{n,\beta n}^z}{n_{\beta n}} < \gamma^2 + \delta_0\right) \\
& \leq \sum_{\substack{\gamma^2 \leq c_1/n_{\beta n} < \gamma^2 + \delta_0, \\ c_2, c_3 \in [4\alpha/\pi(\log K_n)^2, 4/\pi(\log K_n)^2]}} \max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P(K(\tau_{K_n}, x) = c_2, K(\tau_{K_n}, x') = c_3, N_{n,\beta n}^z = c_1) \\
& \leq CK_n^{-2\alpha F_{2,\beta}(\gamma)+\delta},
\end{aligned}$$

and, hence, the desired result holds. \square

6 Proof of Theorem 2.1

We keep using notations introduced at the beginning of the last section.

6.1 Proof of the upper bound in Theorem 2.1

We will prove in this subsection that for any $0 < \alpha, \beta, \delta < 1$, there exist $C > 0$ and $\epsilon > 0$ such that

$$P(|\tilde{\Theta}_{\alpha,0,\beta,n}| \geq K_n^{\rho_2(\alpha,\beta)+4\delta}) \leq CK_n^{-\epsilon}, \quad (36)$$

where

$$\tilde{\Theta}_{\alpha,\beta_2,\beta_1,n} := \{(x, x') \in \tilde{\Psi}_{K_n}(\alpha)^2 : r_{(\beta_2 n - 3) \vee 0} \leq d(x, x') \leq r_{(\beta_1 n - 3) \vee 0}\}.$$

Indeed, (36) implies the upper bound in Theorem 2.1. We begin with observing this.

Note that Theorem 1.1 in [8] provides that for all sufficiently large $n \in \mathbb{N}$,

$$|\Psi_{K_n}(\alpha) \cap \tilde{\Psi}_{K_n}(\alpha)^c| \leq K_n^\delta \quad \text{a.s.} \quad (37)$$

Again, Theorem 1.1 in [8] yields $|\Psi_{K_n}(\alpha)| \leq K_n^{2(1-\alpha)+\delta}$ for all sufficiently large $n \in \mathbb{N}$ a.s. and, hence, (21) yields

$$|\Psi_{K_n}(\alpha)| \leq K_n^{\rho_2(\alpha,\beta)+\delta} \quad \text{a.s.} \quad (38)$$

Then, (37) and (38) yield that for all sufficiently large $n \in \mathbb{N}$,

$$|\{(x, x') : x \in \Psi_{K_n}(\alpha), x' \in \Psi_{K_n}(\alpha) \cap \tilde{\Psi}_{K_n}(\alpha)^c, d(x, x') \leq r_{\beta n - 3}\}| \leq K_n^{\rho_2(\alpha,\beta)+2\delta} \quad \text{a.s.} \quad (39)$$

Note that $\log r_{\beta n-3}/\log K_{n+1} \rightarrow \beta$ and $\beta \mapsto \rho_2(\alpha, \beta)$ is continuous. Hence, if we obtain (36), the symmetry of x and x' , Borel-Cantelli lemma and (39) yield the upper bound in Theorem 2.1.

To show (36), we devide the assertion into two corresponding estimate for $\tilde{\Theta}_{\alpha,0,\beta(1-\alpha),n}$ and $\tilde{\Theta}_{\alpha,\beta(1-\alpha),\beta,n}$. For the former one, note

$$|\tilde{\Theta}_{\alpha,0,\beta(1-\alpha),n}| \leq 4r_{\beta n-3}^2 |\tilde{\Psi}_{K_n}(\alpha)|.$$

Therefore, by (2.18) in [20], (21) yields

$$\begin{aligned} P(|\tilde{\Theta}_{\alpha,0,\beta(1-\alpha),n}| \geq K_n^{\rho_2(\alpha,\beta)+4\delta}) &\leq P(|\tilde{\Psi}_{K_n}(\alpha)| \geq K_n^{2(1-\alpha)+3\delta}) \\ &\leq P(|\Psi_{K_n}(\alpha)| \geq K_n^{2(1-\alpha)+3\delta}) \leq CK_n^{-\epsilon}. \end{aligned} \quad (40)$$

For the latter one, we use the following lemma.

Lemma 6.1. *There exist $h < 2$, $C > 0$ and $\epsilon > 0$ such that for any $\beta' \in [\beta(1-\alpha), \beta]$*

$$q_{n,\beta'} := P(|\tilde{\Theta}_{\alpha,\beta'h/2,\beta',n}| \geq K_n^{\rho_2(\alpha,\beta')+4\delta}) \leq CK_n^{-\epsilon}.$$

We observe that the assertion for $\tilde{\Theta}_{\alpha,\beta(1-\alpha),\beta,n}$ indeed follows from this lemma. Set $\beta_j = \beta(h/2)^j$ and $l = \min\{j : \beta_j \leq \beta(1-\alpha)\}$. By Lemma 6.1 we have $q_{n,\beta_j} \rightarrow 0$ as $n \rightarrow \infty$ for $j = 0, \dots, l-1$. Combining this with the monotonicity of $\beta \mapsto \rho_2(\alpha, \beta)$, we obtain the desired assertion. By virtue of this and (40), we establish (36). Thus, the proof of the upper bound is now reduced to show Lemma 6.1.

Proof of Lemma 6.1. Let $\hat{Z}_{n,\beta'} := 4r_{\beta'n-4}\mathbb{Z}^2 \cap D(0, K_n)$ and $z_{\beta'}(x)$ the point closest to x in $\hat{Z}_{n,\beta'}$. Let δ_0 be as in Corollary 5.1. By the simple argument, we have

$$q_{n,\beta'} \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= P\left(\max_{z \in \hat{Z}_{n,\beta'}} \frac{N_{n,\beta'n}^z}{n_{\beta'n}} \geq \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right), \\ I_2 &:= P\left(|\tilde{\Theta}_{\alpha,\beta'h/2,\beta',n}| \geq K_n^{\rho_2(\alpha,\beta')+4\delta}; \max_{z \in \hat{Z}_{n,\beta'}} \frac{N_{n,\beta'n}^z}{n_{\beta'n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right). \end{aligned}$$

Then,

$$\begin{aligned} I_2 &\leq P\left(\left|\left\{(x, x') \in \tilde{\Psi}_{K_n}(\alpha)^2 : r_{\beta'hn/2-3} \leq d(x, x') \leq r_{\beta'n-3}, \frac{N_{n,\beta'n}^{z_{\beta'}(x)}}{n_{\beta'n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right\}\right| \geq K_n^{\rho_2(\alpha,\beta')+4\delta}\right) \\ &\leq K_n^{-\rho_2(\alpha,\beta')-4\delta} E\left[\left|\left\{(x, x') \in \tilde{\Psi}_{K_n}(\alpha)^2 : r_{\beta'hn/2-3} \leq d(x, x') \leq r_{\beta'n-3}, \frac{N_{n,\beta'n}^{z_{\beta'}(x)}}{n_{\beta'n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right\}\right|\right] \\ &\leq K_n^{-\rho_2(\alpha,\beta')-4\delta} \sum_{\substack{x \in D(0, K_n), \\ x' \in D(x, r_{\beta'n-3}) \cap D(x, r_{\beta'hn/2-3})^c}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha); \frac{N_{n,\beta'n}^{z_{\beta'}(x)}}{n_{\beta'n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right) \\ &\leq CK_n^{-2\alpha F_{2,\beta'}(\min\{1/\sqrt{\alpha}, 2/(2-\beta')\})-3\delta} \max_{\substack{x \in D(0, K_n), \\ x' \in D(x, r_{\beta'n-3}) \cap D(x, r_{\beta'hn/2-3})^c}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha); \frac{N_{n,\beta'n}^{z_{\beta'}(x)}}{n_{\beta'n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0\right)^2\right) \\ &\leq CK_n^{-\delta}. \end{aligned}$$

The last inequality comes from Corollary 5.1. Set $\epsilon := \alpha F_{0,\beta'}(1/\sqrt{\alpha} + \delta_0) - (1 - \beta')$. Note that $\epsilon > 0$. Then, there exists $C > 0$ such that

$$\begin{aligned} I_1 &\leq |\hat{Z}_{n,\beta'}| C K_n^{-2\alpha F_{0,\beta'}(1/\sqrt{\alpha} + \delta_0) + \epsilon/2} \\ &\leq C K_n^{2-2\beta'-2\alpha F_{0,\beta'}(1/\sqrt{\alpha} + \delta_0) + \epsilon} \leq C K_n^{-\epsilon}. \end{aligned}$$

This completes the proof of Lemma 6.1. \square

6.2 Proof of the lower bound in Theorem 2.1

To prove the lower bound in Theorem 2.1, let $r_{n,k} := r_n/r_k$ and

$$\Theta_{\alpha,\beta,n,n'} := \left\{ (x, x') \in \Psi_{K_{n'}}(\alpha)^2 : d(x, x') \leq \frac{r_{n,(1-\beta)n}}{2} \right\}.$$

Fix $\delta > 0$, $0 < \alpha, \beta < 1$ and $\gamma > 0$ with $\delta < \alpha$ and

$$2 - 2\beta - 2\alpha F_{0,\beta}(\gamma) > 2\delta. \quad (41)$$

Our main goal in this section is to show that there exist $\epsilon > 0$ and $C > 0$ such that for any $n \in \mathbb{N}$

$$P(|\Theta_{\alpha-\delta,\beta,n,n+1}| \leq K_n^{2+2\beta-2\alpha F_{2,\beta}(\gamma)-5\delta}) \leq C e^{-\epsilon n}. \quad (42)$$

Note that $\log r_{n,(1-\beta)n+1}/\log K_{n+1} \rightarrow \beta$ and $(\alpha, \beta) \mapsto \rho_2(\alpha, \beta)$ is continuous. Hence, if we obtain (42), the Borel-Cantelli lemma yields the lower bound in Theorem 2.1.

Let \mathcal{A}_n be a maximal set of points in $D(0, 4r_n) \cap D(0, 3r_n)^c$ which are $4r_{n,(1-\beta)n-4}$ separated. The primary idea of the proof of 42 is to consider $\Theta_{\alpha-\delta,\beta,n,n+1} \cap D(z, r_{n,(1-\beta)n}/2)^2$ for each $z \in \mathcal{A}_n$ instead of the whole $\Theta_{\alpha-\delta,\beta,n,n+1}$ and to count the number of elements in these sets. By the choice of the radius, if both $x, x' \in D(z, r_{n,(1-\beta)n}/2)$ are $(\alpha - \delta)$ -favorite points, then $(x, x') \in \Theta_{\alpha-\delta,\beta,n,n+1}$ automatically holds.

As the secondary idea, we restrict random sets to smaller one in order to make the variance of the number of those sets smaller, as explained in introduction. The restriction we introduce here is to control the number of visits to $D(z, r_{n,(1-\beta)n}/2)$ by the number of excursions passing through annuli around $D(z, r_{n,(1-\beta)n}/2)$.

For $z \in \mathcal{A}_n$, $x \in D(z, r_{n,(1-\beta)n}/2)$ and $1 \leq l \leq (1 - \beta)n$, let $\hat{N}_{n,l}^z$ denote the number of excursions from $\partial D(z, r_{n,l})$ to $\partial D(z, r_{n,l-1})$ until time $T_{\partial D(z, r_n)}$. Let $\hat{\mathcal{R}}_{k,m}^z = \hat{\mathcal{R}}_k^z(m)$ be the time until completion of the first m excursions from $\partial D(z, r_{n,k})$ to $\partial D(z, r_{n,k-1})$. Let

$$\tilde{W}_m^z = \tilde{W}^z(m) := \left| \left\{ y \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : K(m, y) \geq \frac{4\alpha}{\pi}(\log K_n)^2 \right\} \right|$$

and

$$\hat{H}_{k,m}^z := \{\tilde{W}_{\hat{\mathcal{R}}_{k,m}^z}^z \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-2\delta}\}.$$

Especially, we let

$$H_k^z := \{\tilde{W}^z(\hat{\mathcal{R}}_k^z(\hat{N}_{n,k}^z)) \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-2\delta}\}. \quad (43)$$

Roughly speaking, the event H_k^z describes the situation that sufficiently many points as $(\alpha - \delta)$ -favortite in a neighborhood of z when all the visit to the neighborhood before leaving $D(z, r_n)$ is finished. We will say that a point $z \in \mathcal{A}_n$ is (n, β) -successful if $|\hat{N}_{n,k}^z - \tilde{n}_k| \leq k$ for all $3 \leq k \leq (1 - \beta)n$ and $H_{n-\beta n}^z$ occurs. That is, the first condition gives a restriction on the number of excursions to a “typical” one.

Indeed, definition comes from the fact which is written in Remark 3.2 in [1]. Note that, this definition is slightly different from that of [7].

By using the notion of succesful points, we can reduce our problem to that for the number of succesful points. Indeed, by the definition of these notations,

$$\begin{aligned} |\Theta_{\alpha-\delta,\beta,n,n}| &\geq \sum_{z \in \mathcal{A}_n} (\tilde{W}^z(\hat{\mathcal{R}}_{n-\beta n}^z(\hat{N}_{n,(1-\beta)n}^z)))^2 \\ &\geq |\{z \in \mathcal{A}_n : z \text{ is } (n, \beta) - \text{successful}\}| K_n^{4\beta-4\alpha(1-\gamma(1-\beta))^2/\beta-4\delta}. \end{aligned}$$

Then, if there exists $C \in (0, 1)$ such that

$$P(|\{z \in \mathcal{A}_n : z \text{ is } (n, \beta) - \text{successful}\}| \geq K_n^{2(1-\beta)-2\gamma^2\alpha(1-\beta)-\delta}) \geq C, \quad (44)$$

we have

$$P(|\Theta_{\alpha-\delta,\beta,n,n}| \leq K_n^{2+2\beta-2\alpha F_{2,\beta}(\gamma)-5\delta}) \leq 1 - C.$$

Since we have $K_{n+1}/K_n \geq n^3$ for $n \in \mathbb{N}$, by the same argument as that after (3.4) in [20] we have

$$P(|\Theta_{\alpha-\delta,\beta,n,n+1}| \leq K_n^{2+2\beta-2\alpha F_{2,\beta}(\gamma)-5\delta}) \leq P(|\Theta_{\alpha-\delta,\beta,n,n}| \leq K_n^{2+2\beta-2\alpha F_{2,\beta}(\gamma)-5\delta}) n^3/2,$$

and, hence, we obtain (42). Thus it suffices to show (44) to conclude the lower bound.

According to the definition of succesful point, we divide the major part of the problem into an estimate of the probability of the event $H_{(1-\beta)n}^z$ and study of the effect of restriction on the number of excursions. Lemma 6.2 concerns with the former one. The latter one will be treated in Lemma 6.4 below in combination with Lemma 6.2. Set $\tilde{n}_k := 6\gamma^2\alpha k^2 \log k$.

Lemma 6.2. *It holds that, uniformly in $z \in \mathcal{A}_n$*

$$P(\hat{H}_{n-\beta n, \tilde{n}_{(1-\beta)n}-(1-\beta)n}^z) = 1 - o(1_n).$$

To prove the Lemma 6.2, we show the following three propositions. To state them we prepare some notation. For $z \in \mathcal{A}_n$, $x \in D(z, r_{n,(1-\beta)n}/2)$ and $(1-\beta)n + 2 \leq l \leq n$, $\hat{N}_{n,l}^{z,x}$ the number of excursions from $\partial D(x, r_{n,l})$ to $\partial D(x, r_{n,l-1})$ until time $\hat{\mathcal{R}}_{(1-\beta)n, \tilde{n}_{(1-\beta)n}-(1-\beta)n}^z$. For $(1-\beta)n \leq l \leq n$, set

$$\hat{n}_l := 6\alpha \left(\frac{l - (1-\beta)n}{\beta} + \left(n - \frac{l - (1-\beta)n}{\beta} \right) \gamma(1-\beta) \right)^2 \log l.$$

For $z \in \mathcal{A}_n$ and $\eta \in \mathbb{N} \cap \{1\}^c$ we say that $x \in D(z, r_{n,(1-\beta)n}/2)$ is (n, β) -qualified if

$$|\hat{N}_{n,l}^{z,x} - \hat{n}_l| \leq n \text{ for } (1-\beta)n + \eta \leq l \leq n.$$

After this, we consider $n \in \mathbb{N}$ with $(1-\beta)n \geq \eta/2$.

Proposition 6.1. *It holds that as $n \rightarrow \infty$,*

$$\begin{aligned} &P\left(\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n, \beta)\text{-qualified}\right\}\right. \\ &\quad \left.\subset \left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : \frac{K(\hat{\mathcal{R}}_{(1-\beta)n, \tilde{n}_{(1-\beta)n}-(1-\beta)n}^z, x)}{(\log K_n)^2} \geq \frac{4\gamma^2\alpha}{\pi} - \frac{2}{\log \log n}\right\}\right) \rightarrow 1. \end{aligned}$$

Proposition 6.2. *For $x \in D(z, r_{n,(1-\beta)n}/2)$,*

$$P(x \text{ is } (n, \beta)\text{-qualified}) = (1 + o(1_n))q_n,$$

where q_n satisfies the following: there exist $c_1(\gamma, \alpha, \beta, \eta)$ and $c_2(\gamma, \alpha, \beta, \eta) > 0$ such that for all sufficiently large $n \in \mathbb{N}$,

$$e^{-c_1 n \log \log n} K_n^{-2\alpha(1-\gamma(1-\beta))^2/\beta} \leq q_n \leq e^{-c_2 n \log \log n} K_n^{-2\alpha(1-\gamma(1-\beta))^2/\beta}.$$

Proposition 6.3. For $x, y \in D(z, r_{n, (1-\beta)n}/2)$, set

$$l(x, y) := \min\{l : D(x, r_{n, l} + 1) \cap D(y, r_{n, l} + 1) = \emptyset\} \wedge n.$$

(i) There exist $c_3(\gamma, \alpha, \beta)$ and $c_4(\gamma, \alpha, \beta) > 0$ such that for sufficiently large $\eta \in \mathbb{N}$, all $x, y \in D(z, r_{n, (1-\beta)n}/2)$ with $(1 - \beta)n + \eta \leq l(x, y) \leq n$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} & P(x \text{ and } y \text{ are } (n, \beta)\text{-qualified}) \\ & \leq n^{c_3} e^{c_4(l-n+\beta n-\eta+1) \log \log n} q_n^2 \exp\left(\frac{6\alpha}{\beta}(1 - \gamma(1 - \beta))^2(l - n + \beta n - \eta + 1) \log n\right). \end{aligned}$$

(ii) For all $x, y \in D(z, r_{n, (1-\beta)n}/2)$ with $l(x, y) \leq (1 - \beta)n + \eta - 1$

$$P(x \text{ and } y \text{ are } (n, \beta)\text{-qualified}) = (1 + o(1_n))q_n^2.$$

Remark 6.1. Note that the choice of sufficiently large $\eta \in \mathbb{N}$ is used instead of the fact that sufficiently large $\bar{\gamma} > 0$ is picked in [1] or [10].

Roughly speaking, Proposition 6.1 ensures that the set of qualified point is essentially a restriction of $\tilde{W}_{\tilde{n}_{(1-\beta)n} - (1-\beta)n}^z$. Propositions 6.2 and 6.3 will be used to apply the moment method based on the Paley-Zygmund inequality.

Proof of Proposition 6.1. We refer to the proof of Lemma 3.1 in [20]. (See also Proposition 3.3 in [1].) Note that for any $x \in D(z, r_{n, (1-\beta)n}/2)$

$$\begin{aligned} & \left\{x \text{ is } (n, \beta)\text{-qualified}\right\} \cap \left\{\frac{K(\hat{\mathcal{R}}_{(1-\beta)n, \tilde{n}_{(1-\beta)n} - (1-\beta)n}^z, x)}{(\log K_n)^2} \geq \frac{4\gamma^2\alpha}{\pi} - \frac{2}{\log \log n}\right\}^c \\ & \subset \left\{K(\hat{\mathcal{R}}_{n, \tilde{n}_n - n}^x, x) \leq \left(\frac{4\gamma^2\alpha}{\pi} - \frac{1}{\log \log n}\right)(\log K_n)^2\right\}. \end{aligned}$$

Since there exists $c > 0$ $|D(0, K_n)| \leq e^{cn \log n}$ for all sufficiently large $n \in \mathbb{N}$, it is sufficient to prove that there exists $c > 0$ such that

$$P\left(K(\hat{\mathcal{R}}_{n, \tilde{n}_n - n}^x, x) \leq \left(\frac{4\gamma^2\alpha}{\pi} - \frac{1}{\log \log n}\right)(\log K_n)^2\right) \leq \exp(-cn(\log n / \log_2 n)^2).$$

This proof is same as that of Lemma 3.1 in [20]. □

Proof of Proposition 6.2. Fix $x \in D(z, r_{n, (1-\beta)n}/2)$. Note that by (6), we have for $(1 - \beta)n + \eta \leq l \leq n - 1$, $x_0 \in \partial D(x, r_{n, l})$,

$$P^{x_0}(T_{\partial D(x, r_{n, l+1})} < T_{\partial D(x, r_{n, l-1})}) = \frac{1}{2}(1 + O(n^{-8}))$$

as well as (4.14) in [20]. Then, for $x_0 \in \partial D(x, r_{n, (1-\beta)n+\eta-1})$,

$$P^{x_0}(T_{\partial D(z, r_{n, (1-\beta)n-1})} < T_{\partial D(x, r_{n, (1-\beta)n+\eta})}) = (1 + O(n^{-8}))\frac{1}{\eta + 1}.$$

In addition, for $m_{(1-\beta)n+\eta}$ with $|m_{(1-\beta)n+\eta} - \hat{n}_{(1-\beta)n+\eta}| \leq n$, let

$$\begin{aligned} \hat{q}_n := & \sum_{1 \leq i \leq m_{(1-\beta)n+\eta} \wedge (\tilde{n}_{(1-\beta)n} - (1-\beta)n - 1)} \left(\frac{\eta}{\eta + 1}\right)^{\tilde{n}_{(1-\beta)n} - (1-\beta)n - 1 - i} \left(\frac{1}{\eta + 1}\right)^i \frac{(\tilde{n}_{(1-\beta)n} - (1 - \beta)n - 1)!}{(\tilde{n}_{(1-\beta)n} - (1 - \beta)n - 1 - i)!i!} \\ & \left(\frac{1}{\eta + 1}\right)^i \left(\frac{\eta}{\eta + 1}\right)^{m_{(1-\beta)n+\eta} - i} \frac{(m_{(1-\beta)n+\eta} - 1)!}{(m_{(1-\beta)n+\eta} - 1 - i)!i!}. \end{aligned}$$

Then, (10) and the same argument as Proposition 3.4 in [1], (4.13) in [20] or (5.9) in [10] yield that

$$P(x \text{ is } (n, \beta) - \text{qualified}) = \sum P(|\hat{N}_{n,l}^{z,x} - \hat{n}_l| \leq n \text{ for } (1-\beta)n + \eta \leq l \leq n) = (1 + o(1_n))q_n, \quad (45)$$

where

$$q_n := \sum \hat{q}_n \times \prod_{l=(1-\beta)n+\eta+1}^n \binom{m_l + m_{l-1} - 1}{m_l} \left(\frac{1}{2}\right)^{m_l + m_{l-1} - 1}. \quad (46)$$

Here, the summations in (45) and (46) are over all $m_{(1-\beta)n+\eta}, \dots, m_n$ with $|m_i - \hat{n}_i| \leq n$ for $(1-\beta)n + \eta \leq l \leq n$. Hence, the Stirling formula yields that for $m_{(1-\beta)n+\eta}$ with $|m_{(1-\beta)n+\eta} - \hat{n}_{(1-\beta)n+\eta}| \leq n$,

$$\begin{aligned} \hat{q}_n &\geq \left(\frac{\eta}{\eta+1}\right)^{\tilde{n}_{\beta n} - \beta n - 1 - i} \left(\frac{1}{\eta+1}\right)^i \frac{(\tilde{n}_{\beta n} - \beta n - 1)!}{(\tilde{n}_{\beta n} - \beta n - 1 - i)!i!} \left(\frac{1}{\eta+1}\right)^i \left(\frac{\eta}{\eta+1}\right)^{m_{(1-\beta)n+\eta} - i} \\ &\quad \frac{(m_{(1-\beta)n+\eta} - 1)!}{(m_{(1-\beta)n+\eta} - 1 - i)!i!} \Big|_{i=(2\eta\sqrt{\tilde{n}_{\beta n}m_{(1-\beta)n+\eta}} - (m_{(1-\beta)n+\eta} + \tilde{n}_{\beta n} - \beta n))/2(\eta^2 - 1)} \\ &= n^{o(1)}. \end{aligned} \quad (47)$$

By the same estimate as in the proof of Proposition 3.4 in [1], we have for all m_i with $|m_i - \hat{n}_i| \leq n$ and $(1-\beta)n + \eta + 1 \leq l \leq n$,

$$\begin{aligned} \binom{m_l + m_{l-1} - 1}{m_l} \left(\frac{1}{2}\right)^{m_l + m_{l-1} - 1} &\geq c(m_l)^{-1/2} \exp \left\{ -m_{l-1} \left(\frac{1}{4} \left(\frac{m_l}{m_{l-1}} - 1 \right)^2 + c' \left| \frac{m_l}{m_{l-1}} - 1 \right|^3 \right) \right\} \\ &\geq cn^{-1} (\log n)^{-1/2} \exp \left\{ -\frac{6\alpha}{\beta^2} (1 - \gamma(1-\beta))^2 \log n \right\}. \end{aligned} \quad (48)$$

Therefore, we obtain

$$q_n \geq c(\log n)^{-c'n} K_n^{-2\alpha(1-\gamma(1-\beta))^2/\beta},$$

and, hence, the desired lower bound holds. Note that by (10) it is trivial that

$$\sum_{|m_{(1-\beta)n+\eta} - \hat{n}_{(1-\beta)n+\eta}| \leq n} \hat{q}_n \leq 1.$$

If we use it instead of (47) and the similar computation to (48), we have the upper bound. \square

In order to prove Proposition 6.3, we set the following sigma field as [1, 10]. First, define a sequence of stopping times. Fix $z \in \mathcal{A}_n$ and $x \in D(z, r_{n,(1-\beta)n}/2)$. For $(1-\beta)n + \eta \leq l \leq n$, let $\sigma_x^{(0)}[l] := 0$ and

$$\begin{aligned} \tau_x^{(0)}[l] &:= \inf\{m \geq 0 : S_m \in \partial D(x, r_{n,l})\}, \\ \sigma_x^{(i)}[l] &:= \inf\{m \geq \tau_x^{(i-1)}[l] : S_m \in \partial D(x, r_{n,l-1})\} \text{ for } i \geq 1, \\ \tau_x^{(i)}[l] &:= \inf\{m \geq \sigma_x^{(i)}[l] : S_m \in \partial D(x, r_{n,l})\} \text{ for } i \geq 1. \end{aligned}$$

In addition, let $\mathcal{G}_l^x := \sigma(\bigcup_{i \geq 0} \{S_m : \sigma_x^{(i)}[l] \leq m \leq \tau_x^{(i)}[l]\})$. Then, we will obtain the following lemma.

Lemma 6.3. *There exists ϵ_n with $\lim_{n \rightarrow \infty} \epsilon_n = 0$ such that the following holds for all $n - \beta n + \eta \leq l \leq n - 1$, with $|m_l - \hat{n}_l| \leq n$ and $x \in D(z, r_{n,(1-\beta)n}/2)$,*

$$\begin{aligned} &P(\hat{N}_{n,i}^{z,x} = m_i \text{ for all } i = l, \dots, n | \mathcal{G}_l^x) \\ &= (1 + \epsilon_n) P(\hat{N}_{n,i}^{z,x} = m_i \text{ for all } i = l + 1, \dots, n | \hat{N}_{n,l}^{z,x} = m_l) 1_{\{\hat{N}_{n,l}^{z,x} = m_l\}}. \end{aligned}$$

Proof. The proof is the same as that of Corollary 5.1 in [10] since Lemma 2.4 in [10] holds even for the simple random walk in \mathbb{Z}^2 instead of the one in 2-dimensional torus. (See also Lemma 3.6 in [1].) \square

Proof of Proposition 6.3. The proof is almost same as that of Proposition 3.5 in [1] since the same argument as [1] holds even for the discrete-time simple random walk in \mathbb{Z}^2 instead of the continuous-time one with Lemma 6.3. (See also Lemma 3.2 in [20] or Lemma 4.2 in [10].) \square

Proof of Lemma 6.2. As mentioned in Remark 6.1, we consider the same argument as that of proof of the lower bound in Theorem 1.2 (i) in [1] (or [10]). If we pick η with $c_3 < 6\eta - 7$ in Proposition 6.3, Propositions 6.2 and 6.3 yield that for all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} E \left[\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x \text{ is } (n, \beta)\text{-qualified} \right\} \right| \right] &\geq (1 + o(1)) \left| D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) \right| q_n, \\ E \left[\left| \left\{ (x, y) \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x, y \text{ are } (n, \beta)\text{-qualified}, l(x, y) \geq (1 - \beta)n + \eta \right\} \right| \right] \\ &\leq \left| D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) \right|^2 (1 + o(1)) n^{c_3} q_n^2 \sum_{l=(1-\beta)n+\eta}^n n^{-6\eta+7-6(l-n+\beta n-\eta+1)} e^{c_4(l-n+\beta n-\eta+1) \log \log n} \\ &\leq o(1) E \left[\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x \text{ is } (n, \beta)\text{-qualified} \right\} \right| \right]^2 \end{aligned}$$

and

$$\begin{aligned} E \left[\left| \left\{ (x, y) \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x, y \text{ are } (n, \beta)\text{-qualified}, l(x, y) \leq (1 - \beta)n + \eta - 1 \right\} \right| \right] \\ \leq (1 + o(1)) \left| D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) \right|^2 q_n^2. \end{aligned}$$

Then, for uniformly in $z \in \mathcal{A}_n$,

$$\begin{aligned} E \left[\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x \text{ is } (n, \beta)\text{-qualified} \right\} \right|^2 \right] \\ \leq (1 + o(1)) E \left[\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x \text{ is } (n, \beta)\text{-qualified} \right\} \right| \right]^2. \end{aligned}$$

In addition, Proposition 6.2 yields that for all sufficiently large $n \in \mathbb{N}$,

$$E \left[\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x \text{ is } (n, \beta)\text{-qualified} \right\} \right| \right] \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-\delta}.$$

Then, by the Paley-Zygmund inequality, we obtain that as $n \rightarrow \infty$,

$$P \left(\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : x \text{ is } (n, \beta)\text{-qualified} \right\} \right| \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-2\delta} \right) \rightarrow 1.$$

Therefore, by Proposition 6.1, as $n \rightarrow \infty$,

$$\begin{aligned} P \left(\left| \left\{ x \in D \left(z, \frac{r_{n,(1-\beta)n}}{2} \right) : K(\hat{\mathcal{R}}_{(1-\beta)n, \tilde{n}_{(1-\beta)n} - (1-\beta)n}^z, x) \geq \left(\frac{4\gamma^2\alpha}{\pi} - \frac{1}{\log \log n} \right) (\log K_n)^2 \right\} \right| \right. \\ \left. \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-2\delta} \right) \rightarrow 1, \end{aligned}$$

and, hence, Lemma 6.2 holds for all sufficiently large $n \in \mathbb{N}$ with $1/\log_2 n < \delta$. \square

As mentioned above, we will introduce the following lemma in order to prove (44).

Lemma 6.4. *There exists $\delta_n > 0$ with $\lim_{n \rightarrow \infty} \delta_n = 0$ such that*

$$\bar{q}_n := \inf_{x \in \mathcal{A}_n} P(x \text{ is } (n, \beta)\text{-successful}) \geq r_{(1-\beta)n}^{-(2\gamma^2\alpha + \delta_n)}. \quad (49)$$

Further, there exists $c > 0$ such that for all $x \neq y \in \mathcal{A}_n$,

$$P(x \text{ and } y \text{ are } (n, \beta)\text{-successful}) \leq c\bar{q}_n^2 r_{l(x,y)}^{2\gamma^2\alpha + \delta_{l(x,y)}}. \quad (50)$$

Proof. We refer to the proof of Lemma 2.1 in [7] or Lemma 10.1 in [10]. Substitute n , $1 - \beta$, $l(x, y)$, $2\gamma^2\alpha$ into m , β , $k(x, y)$, a in Lemma 2.1 of [7] respectively and consider our event $H_{(1-\beta)n}^x$ and $\sup_{x \in \mathcal{A}_n} P(H_{(1-\beta)n, \tilde{n}_{(1-\beta)n} - (1-\beta)n}^z)$ in (43) instead of $H_{\beta m}^x$ and ξ_m in [7]. Now, we will say that a point $z \in \mathcal{A}_n$ is (n, β) -pre-successful if $|\hat{N}_{n,k}^z - \tilde{n}_k| \leq k$ for all $3 \leq k \leq n - \beta n$. As mentioned in [7], if we replace “ (n, β) -successful” in (49) and (50) by (n, β) -pre-successful, the corresponding assertion is essentially established in Lemma 3.2 in [20], while he considered the case $\beta = 0$ and his $r_{n,k}$ is different from ours. Since (n, β) -successful point is also (n, β) -pre-successful point, this yields (50). In addition, we obtain that, uniformly in $x \in \mathcal{A}_n$,

$$P(x \text{ is } (n, \beta)\text{-successful}) \geq (1 + o(1_n))P(x \text{ is } (n, \beta)\text{-pre-successful}).$$

by using Lemma 6.2 instead of (2.13) in [7]. Therefore, (49) holds. \square

Proof of (44). By the same argument as (2.11) in [7], we have that there exists $C > 0$ such that

$$E[|\{z \in \mathcal{A}_n : z \text{ is } (n, \beta)\text{-successful}\}|^2] \leq CE[|\{z \in \mathcal{A}_n : z \text{ is } (n, \beta)\text{-successful}\}|]^2,$$

by using Lemma 6.4 instead of Lemma 2.1 in [7]. By the Paley-Zygmund inequality, we have (44). \square

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